Polynomial Expansion of a Cumulant

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In a recent paper Wishart[1] has given a differential recursion formula for the cumulants of multinomial multivariate distributions. The object of the present note is to indicate a method of expanding the general cumulant of a distribution, satisfying a simple condition, in a polynomial in lower-order cumulants.

1. The Univariate Case

Let $K(t)$ be the c.g.f. of a distribution and let A denote the operator d/dt, so that

$$K_n = A^n K(0). \tag{1.1}$$

Now

$$A^n K(t) = A^{n-2}\{A^2 K(t)\}, \tag{1.2}$$

so that if $A^2 K(t)$ can be expressed as the product of two linear functions of $AK(t)$, say as

$$A^2 K(t) = \{aAK(t) + b\}\{cAK(t) + d\}, \tag{1.3}$$

we can re-write (1.2) in the form

$$A^n K(t) = A^{n-2}[\{aAK(t) + b\}\{cAK(t) + d\}].$$

By expanding the right-hand side of this equation by Leibniz's theorem and then putting $t = 0$, we get

$$K_n = aK_{n-1}(cK_1 + d) + \ldots + \binom{n-2}{r} aK_{n-r-1}.cK_{r+1} + \ldots + + (aK_1 + b)\, cK_{n-1} \tag{1.4}$$

This is the expansion desired.

Example 1. *The Binomial distribution.*—Here, if s be the number of independent trials and p the basic probability,

$$K(t) = s \log (q + pe^t).$$

Therefore

$$AK(t) = \frac{spe^t}{q + pe^t} = \frac{spe^t}{D}, \text{ say;}$$

$$A^2K(t) = \frac{spe^t}{D} - \frac{spe^t . pe^t}{D^2} = AK(t)\left\{1 - \frac{AK(t)}{s}\right\}.$$

Hence we have

$$K_n = K_{n-1}\left(1 - \frac{K_1}{s}\right) + \ldots + \binom{n-2}{r} K_{n-r-1} \cdot \frac{-K_{r+1}}{s}$$
$$+ \ldots + K_1 \cdot \frac{-K_{n-1}}{s}. \qquad (1.5)$$

Example 2. *The Polya distribution* is defined by

$$P_0 = (1 + \beta\mu)^{-\frac{1}{\beta}} \qquad \beta \geqslant 0,\ \mu > 0;$$
$$P_i = \left(\frac{\mu}{1+\beta\mu}\right)^i \cdot \frac{1\,(1+\beta)\ldots(1+\overline{i-1}\,\beta)}{i}\, P_0,\ i \geqslant 1.$$

The c.g.f. is easily seen to be

$$K(t) = -\frac{1}{\beta}\log(1 + \beta\mu - \beta\mu e^t).$$

Therefore

$$AK(t) = \frac{\mu e^t}{1 + \beta\mu - \beta\mu e^t} = \frac{\mu e^t}{D}, \text{ say};$$

$$A^2K(t) = \frac{\mu e^t}{D} + \frac{\mu e^t \cdot \beta\mu e^t}{D^2} = AK(t)\{1 + \beta AK(t)\}.$$

Hence we get for this distribution

$$K_n = K_{n-1}(1 + \beta K_1) + \ldots + \binom{n-2}{r} K_{n-r-1} \cdot \beta K_{r+1} + \ldots +$$
$$+ K_1 \cdot \beta K_{n-1} \qquad (1.6)$$

The Polya distribution tends to the Poisson or the Pascal distribution according as $\beta \to 0$ or $\beta \to 1$. By taking corresponding limits in (1.6), we thus get the formulæ for the cumulants of these distributions.

2. THE MULTIVARIATE CASE

The extension of the method to the multivariate case is straightforward, and can shortly be indicated as follows:—

Let $K(t, u, v, \ldots)$ be the c.g.f. of a multivariate distribution, so that

$$K_{n_1 . n_2 . n_3 \ldots} = \left[\frac{\partial^{n_1+n_2+n_3+\cdots} K(0, 0, 0, \ldots)}{\partial t^{n_1} \partial u^{n_2} \partial v^{n_3} \ldots}\right];$$

we define as *basic* those cumulants which have every order-suffix equal to 0 or 1. If, now, we can express every cumulant of order 2 in one variate and orders zero or unity in all the other variates as the product of two linear functions of the basic cumulants, we may apply an obvious generalisation of Leibniz's theorem and get the desired polynomial expansion for $K_{n_1 . n_2 . n_3 . \ldots}$. To save space I am giving here the results only for the bivariate Bernoullian and Pascal distributions without any algebra.

Example 3. *The Bivariate Binomial distribution.*—The c.g.f. is

$$K(t, u) = s \log (p_{00} + p_{10} e^t + p_{01} e^u + p_{11} e^{t+u}).$$

The expansion formulæ for $K_{n_1.1}$ and $K_{1.n_2}$ are found to be

$$\left.\begin{aligned} K_{n_1.1} &= K_{n_1-1.1} - \frac{2}{s} \sum_{r=0}^{n_1-2} \binom{n_1-2}{r} K_{n_1-r-1.1} K_{r+1.0} \\ K_{1.n_2} &= K_{1.n_2-1} - \frac{2}{s} \sum_{r=0}^{n_2-2} \binom{n_2-2}{r} K_{1.n_2-r-1} K_{0.r+1} \end{aligned}\right\} \quad (2.1)$$

The general formula is

$$\left.\begin{aligned} K_{n_1.n_2} &= K_{n_1-1.n_2} - \frac{2}{s}\left[\sum_{r=0}^{n_1-2} \binom{n_1-2}{r} \left\{\sum_{t=0}^{n_2-1} \binom{n_2-1}{t} K_{n_1-r-1.n_2-t} \right.\right. \\ &\qquad \left.\left. . K_{r+1.t}\right\}\right] \\ &= K_{n_1.n_2-1} - \frac{2}{s}\left[\sum_{r=0}^{n_2-2} \binom{n_2-2}{r} \left\{\sum_{t=0}^{n_1-2} \binom{n_1-1}{t} K_{n_1-t.n_2-r-1} \right.\right. \\ &\qquad \left.\left. . K_{t.r+1}\right\}\right] \end{aligned}\right\} \quad (2.2)$$

Example 4. For the *Bivariate Negative Binomial* or *Pascal distribution*, the formulæ are the same as (2.1) and (2.2) except that all signs are positive in this case.

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REFERENCE

. Wishart, J. .. *Biometrika*, 36, pp. 47–58.